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THE MODIFIED KUHN MODEL OF LINEAR VISCOELASTICITY

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Abstract—A model of linear viscoelasticity, applicable to the description of structural materials whose creep is asymptotically logarithmic, is derived on the basis of some substantial modifications of a model introduced by Kuhn *et al.* (1947, *Helv. Chim. Acta* **30**, 307–328, 464–486) to describe rubber. From the proposed creep function, the relaxation function is determined by Laplace transform inversion. Energy dissipation is studied, and in particular the frequency dependence of the loss tangent, which is found to be flat over a large number of cycles if the damping is small, in accordance with observed behavior of structural materials and unlike the usual (Kelvin) model of structural damping. An approximation to the modified model based on a discrete retardation-time spectrum, called the discrete modified Kuhn model, is also studied; this is an internal-variable model which can be extended to account for nonlinear behavior. In contrast to existing rheological models, in the proposed model the number of parameters is fixed and does not depend on the number of rheological elements needed to simulate experimental results.

1. INTRODUCTION

Beginning with the experiments by Phillips (1905), the creep of numerous solids (including rubber, glass, various metals and concrete) has been observed to obey a logarithmic creep law over a wide range of times; that is, at a time t sufficiently long after the application of stress, the strain is given approximately by

$$\varepsilon \propto \ln t. \tag{1}$$

For concrete in the linear range, for example, the creep function

$$J(t) = A + B \log(1 + t/\tau), \tag{2}$$

is often used (U.S. Bureau of Reclamation, 1956) to describe creep over a time range of months or years if τ is a characteristic time of the order of one day. Obviously, eqn (2) reduces to the form (1) for t sufficiently large.

Following Phillips, the creep of rubber was traditionally described by the creep function

$$J(t) = A + B \ln t \tag{3}$$

and its time derivative

$$\dot{J}(t) = \frac{B}{t}. \tag{4}$$

In order to remedy the singularity at $t = 0$, Kuhn *et al.* (1947) proposed to replace the expression (4) for the time derivative of the creep function with

$$\dot{J}(t) = \frac{B}{t}(1 - e^{-Ct}), \tag{5}$$

which has the finite limit BC as $t \to 0$ and has the same behavior as (4) as $t \gg 1/C$. Kuhn *et al.* defined the creep function as the integral of (5) from 0 to t, with no constant of integration, consistent with the fact that the instantaneous deformation of rubber, based on "glassy" response [i.e. for temperatures below the glass-transition temperature T_g; see e.g. DiBenedetto (1967) or Van Vlack (1989)], is negligible when compared with later deformation (based on "rubbery" response, i.e. for temperatures above T_g). This creep function can be expressed in terms of the exponential integral [see e.g. Greenberg (1978)] as follows:

$$J(t) = B\int_0^t \frac{1-e^{-Cx}}{x}\,dx = B[\gamma+\ln(Ct)+\mathrm{Ei}\,(Ct)], \tag{6}$$

where Ei represents the exponential integral and $\gamma = 0.5772\ldots$ is Euler's constant. It is seen from this expression that $J(0) = 0$ and that the logarithmic feature of the observed creep function (3) is preserved.

It is the purpose of the present paper to propose several modifications to the Kuhn creep function (6) that will make it more readily applicable to solids such as concrete (in which the instantaneous strain is not negligible beside the creep strain) and generalizable to multiaxial stress states and to nonlinear behavior.

2. THE CONTINUOUS MODIFIED KUHN MODEL

2.1. *Creep function*

The first modification, at first glance minor but in fact significant in that it makes possible the description of structural materials, is the inclusion of a nonzero constant of integration in the expression for the creep function in order to account for instantaneous elasticity. The creep function is therefore

$$J(t) = A+B\int_0^t \frac{1-e^{-Cx}}{x}\,dx, \tag{7}$$

where A is the instantaneous elastic compliance.

The second modification involves a change of variable, namely $Cx = t/\tau$, where τ is the new dummy variable, in order to obtain a rheological representation of the model. The creep function can then be written in the following retardation-time superposition form:

$$J(t) = A+B\int_{1/C}^{\infty} (1-e^{-t/\tau})\frac{d\tau}{\tau}. \tag{8}$$

This creep function given by (7) or (8) will be said to represent the *continuous modified Kuhn model*, since it is based on a continuous retardation-time spectrum, in order to distinguish it from the *discrete modified Kuhn model* to be discussed in Section 4.

2.2. *Relaxation function*

The relaxation function $G(t)$ corresponding to the creep function (8) may be determined by means of Laplace transform methods. It is well known [see e.g. Christensen (1982) or Flügge (1975)] that the Laplace transforms of the creep and relaxation functions, written respectively as $\bar{J}(s) \stackrel{\text{def}}{=} \mathscr{L}\{J(t)\}$ and $\bar{G}(s) \stackrel{\text{def}}{=} \mathscr{L}\{G(t)\}$ (where s is the transform variable, in general complex), are related by

$$\bar{J}(s)\bar{G}(s) = \frac{1}{s^2}. \tag{9}$$

The relaxation function may therefore be found by inverting $\bar{G}(s)$ as given by (9). Using the creep function in the form given by eqn (7), we have

$$\bar{J}(s) = \frac{A}{s} + B\int_0^t \left(\int_0^t \frac{1-e^{-C\tau}}{\tau}\,d\tau\right) e^{-st}\,dt.$$

or

$$\bar{J}(s) = \frac{A}{s} + B\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{A}{s} + B\frac{\bar{f}(s)}{s}, \tag{10}$$

where

$$f(t) \stackrel{\text{def}}{=} \frac{1-e^{-Ct}}{t},$$

so that

$$\bar{f}(s) = \mathscr{L}\{f(t)\} = \mathscr{L}\left\{\frac{g(t)}{t}\right\},$$

where

$$g(t) = 1 - e^{-Ct}.$$

From the properties of the Laplace transform it follows that

$$\bar{f}(s) = \int_s^\infty \bar{g}(u)\,du,$$

while

$$\bar{g}(s) = \int_0^\infty (1-e^{-Ct})\,e^{-su}\,du = \frac{1}{s} - \frac{1}{C+s}.$$

Therefore

$$\bar{f}(s) = \int_s^\infty \left(\frac{1}{u} - \frac{1}{u+C}\right) du = \ln\frac{s+C}{s}.$$

As a result of eqn (10), the Laplace transform of the creep function is

$$\bar{J}(s) = \frac{1}{s}\left[A + B\ln\left(1+\frac{C}{s}\right)\right], \tag{11}$$

and by relation (9) the Laplace transform of the relaxation function is found to be

$$\bar{G}(s) = \frac{1}{s\left[A + B\ln\left(1+\frac{C}{s}\right)\right]}.$$

It follows from the Laplace transform inversion theorem that the relaxation function $G(t)$ is given by the Bromwich integral:

$$G(t) = \mathscr{L}^{-1}[\bar{G}(s)] = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \bar{G}(s)\,e^{st}\,ds,$$

where α denotes an abscissa located to the right of any singularity $\bar{G}(s)$ may have. Hence,

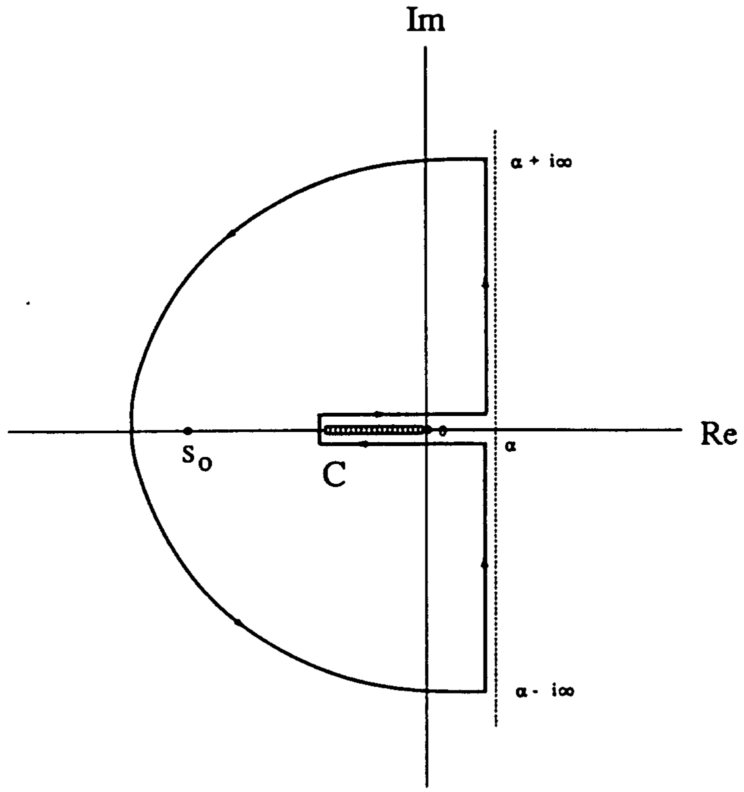

Fig. 1. Singularities of $\bar{G}(s)$.

$$G(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{e^{st}}{s\left[A + B\ln\left(1 + \dfrac{C}{s}\right)\right]}\,ds. \tag{12}$$

The integrand must be examined in order to determine its singularities, and in particular poles and branch cuts. The singularity at $s = 0$ is removable. On the other hand, the one at $s = s_0$, where $A + B\ln(1 + C/s_0) = 0$, is a pole. We may solve for s_0, obtaining

$$s_0 = -\frac{C}{1 - e^{-A/B}}. \tag{13}$$

Finally, there is a logarithmic branch cut on the negative real axis from $-C$ to 0, since for s in the interval $[-C, 0]$, $1 + C/s < 0$. It is observed from eqn (13) that $s_0 < -C$. Hence the description in the complex plane is the one portrayed in Fig. 1. With the aid of Cauchy's integral theorem, the relaxation function $G(t)$ given by eqn (12) becomes

$$G(t) = \mathrm{Res}\,(e^{st}\,\bar{G}(s), s_0) + \frac{1}{2\pi i}(I_1 + I_2), \tag{14}$$

where $\mathrm{Res}\,(e^{st}\,\bar{G}(s), s_0)$ is the residue of the function $e^{st}\,\bar{G}(s)$ associated with the pole $s = s_0$, and I_1 and I_2 are the integrals above and below the branch cut, respectively. Using the standard method in complex analysis [see e.g. Marsden and Hoffman (1987)], we found the residue to be

$$\operatorname{Res}(e^{st}\bar{G}, s_0) = -\frac{e^{-Ct/(1-e^{-A/B})}}{B(e^{A/B}-1)}. \tag{15}$$

In order to evaluate the integrals I_1 and I_2, we note that above the branch cut $s = r\,e^{i\pi}$, where $0 < r < C$, so that

$$\ln\left(1+\frac{C}{s}\right) = \ln\left(\frac{C}{r}-1\right)+i\pi,$$

while below the branch cut $s = r\,e^{-i\pi}$, and hence

$$\ln\left(1+\frac{C}{s}\right) = \ln\left(\frac{C}{r}-1\right)-i\pi.$$

With the help of these expressions, we obtain

$$\frac{1}{2\pi i}(I_1+I_2) = \frac{1}{\pi}\operatorname{Im}\int_0^C \frac{e^{-rt}}{r[A+B\ln(C/r-1)-iB\pi]}\,dr,$$

where Im denotes the imaginary part of a complex quantity. By performing the change of variables $x = r/C$, and upon noting that $0 < r < C$ implies $0 < x < 1$, we may transform this integral into

$$\frac{1}{2\pi i}(I_1+I_2) = B\int_0^1 \frac{e^{-xCt}}{x\{[A+B\ln(1/x-1)]^2+(B\pi)^2\}}\,dx.$$

Inserting this result, along with eqn (15), into (14), we obtain the relaxation function in a form in which it may be evaluated by numerical integration, namely

$$G(t) = -\frac{e^{-Ct/(1-e^{-A/B})}}{B(e^{A/B}-1)} + B\int_0^1 \frac{e^{-xCt}}{x\{[A+B\ln(1/x-1)]^2+(B\pi)^2\}}\,dx.$$

While the integrand blows up at $x = 0$, it can be shown that the integral is well behaved there. For sufficiently small x, the denominator is approximately $B^2x[\ln(1/x)]^2$, so that it is sufficient to examine the behavior of the integral

$$\int_0^h \frac{dx}{x[\ln(1/x)]^2} = \int_0^h \frac{dx}{x(\ln x)^2}$$

for some small h ($h \ll 1$). By introducing the variable $u = \ln x$, this integral becomes

$$\int_{-\infty}^{\ln h} \frac{du}{u^2} = \frac{1}{\ln(1/h)}.$$

3. ENERGY DISSIPATION

3.1. *Thermodynamic considerations*

Energy dissipation in a linearly viscoelastic material may be discussed on the basis of the thermodynamics of materials with internal variables. In this section special attention will be paid to the description of the dissipation by the proposed model, in particular its relation with the frequency of excitation. By way of background it is recalled that if an oscillatory strain of angular frequency ω,

$$\varepsilon(t) = \varepsilon_0 \sin \omega t, \tag{16}$$

is applied to a linear viscoelastic material specimen, the steady-state stress response σ is also oscillatory with the same angular frequency ω, but out of phase with the strain. In particular, the stress leads the strain by a phase angle $\delta(\omega)$ and is given by

$$\sigma(t) = \sigma_0 \sin(\omega t + \delta(\omega)). \tag{17}$$

For simplicity, the explicit dependence of the angle δ on ω will henceforth be omitted. The dissipated energy, as will be seen, is associated with the phase angle δ. If ωt is eliminated between eqns (16)–(17), the following relation is obtained:

$$\frac{\sigma^2(t)}{\sigma_0^2} + \frac{\varepsilon^2(t)}{\varepsilon_0^2} - 2\sigma(t)\varepsilon(t)\frac{\cos\delta}{\varepsilon_0\sigma_0} = \sin^2\delta.$$

This equation describes an ellipse in the σ–ε plane.

The dissipated energy may be calculated by appealing to the first and second laws of thermodynamics. The local form of this law is

$$\rho \dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \rho r - \operatorname{div} \mathbf{h}, \tag{18}$$

[see e.g. Lubliner (1990)], where u is the internal energy density, ρ is the material density, r is the rate of body heating or radiation per unit mass, and $\mathbf{h}$ is the heat-flux vector (so that $h = \mathbf{h}\cdot\mathbf{n}$ is the heat outflow per unit time per unit surface with normal vector $\mathbf{n}$). The inner product of the stress with the strain rate tensor represents the mechanical power (also called deformation power or stress power) per unit volume.

Let a *cyclic process* in a material element be defined as a process in which the kinematic and response functions defining the state of the element have the same values at the beginning (time t_1) and at the end of the process (time t_2). Upon integrating eqn (18) from t_1 to t_2, it is seen that

$$\int_{t_1}^{t_2} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}\, dt + \int_{t_1}^{t_2} (\rho r - \operatorname{div} \mathbf{h})\, dt = 0,$$

since the internal-energy density is a state function and therefore has the same value at instants at which the state is the same. Since the steady-state excitation is a cyclic process, it follows that the integral of the deformation power over a period (equal to the area inside the stress–strain loop in the one-dimensional case) equals the amount of heat produced (or the mechanical energy dissipated) during this time. It is clear from the above derivation that the area of the stress–strain loop represents the dissipated energy only in the case of a steady-state deformation and for a period. It can be mentioned here that Tschoegl (1989) arrives at the same result on the basis of the assumption that the kinetic energy can be neglected for a viscoelastic material; this assumption is unnecessary, and is wrong in general.

The second law of thermodynamics may be expressed by the Kelvin inequality

$$D = \sum_\alpha p_\alpha \dot{\xi}_\alpha \geqslant 0,$$

(Lubliner, 1990), where D is the intrinsic dissipation (Lemaitre and Chaboche, 1990),

$$p_\alpha = -\rho \frac{\partial \psi}{\partial \xi_\alpha}$$

is the "thermodynamic force" conjugate to the internal variable ξ_α, and ψ is the Helmholtz free-energy density per unit mass. It was shown by Lubliner (1972) that the additive decomposition of the strain tensor $\boldsymbol{\varepsilon}$ into elastic ($\boldsymbol{\varepsilon}^e$) and inelastic parts ($\boldsymbol{\varepsilon}^i$) is compatible with the existence of a free energy density $\psi(\boldsymbol{\varepsilon}, \boldsymbol{\xi}, T)$ if and only if the free energy can be decomposed as follows:

$$\psi(\boldsymbol{\varepsilon}, T, \boldsymbol{\xi}) = \psi^e(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^i(\boldsymbol{\xi}), T)+\psi^i(\boldsymbol{\xi}, T). \tag{19}$$

Then the dissipation becomes

$$D = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^i - \rho \sum_\alpha \frac{\partial \psi^i}{\partial \xi_\alpha} \dot{\xi}_\alpha$$

with the help of the standard relation $\boldsymbol{\sigma} = \rho\, \partial\psi/\partial\boldsymbol{\varepsilon}$ [see e.g. Lubliner (1990)].

If the part of the dissipation made up of the inelastic work rate $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^i$ is integrated through a cyclic process, we obtain

$$\int_{t_1}^{t_2} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^i \,\mathrm{d}t = \int_{t_1}^{t_2} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \,\mathrm{d}t - \int_{t_1}^{t_2} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^e \,\mathrm{d}t.$$

Because of the relation (19), the stress tensor is given by

$$\boldsymbol{\sigma} = \rho \frac{\partial \psi^e}{\partial \boldsymbol{\varepsilon}^e}.$$

Hence the second integral of the right-hand side equals the difference of the values of ψ^e (which is a function of $\boldsymbol{\varepsilon}^e$ and T) at the beginning and at the end of the cycle and is zero for isothermal conditions [$\boldsymbol{\varepsilon}^e$ attains the same values at the beginning and end of the cycle for isothermal conditions, since $\boldsymbol{\sigma}$ does so and $\boldsymbol{\varepsilon}^e = \hat{\boldsymbol{\varepsilon}}^e(\boldsymbol{\sigma}, T)$]. Since $\boldsymbol{\varepsilon}(\boldsymbol{\sigma}, T, \boldsymbol{\xi})$ and $\boldsymbol{\sigma}$ attain the same values at the end of the cycle and because specifically a steady-state cycle is being considered, it is reasonable to assume that in such a cycle the internal-variable vector $\boldsymbol{\xi}$ also has the same value at the end of the cycle. Hence the integral over a cycle of the second part of the dissipation $D = -\rho \sum_\alpha (\partial\psi^i/\partial\xi_\alpha)\dot{\xi}_\alpha$ equals zero in an isothermal process, in view of eqn (19). Hence

$$W_d = \int_{t_1}^{t_2} D \,\mathrm{d}t = \int_{t_1}^{t_2} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \,\mathrm{d}t.$$

Consequently, we have arrived at the same result as before, i.e. the dissipated energy is given by the area of the stress–strain loop in a steady-state cycle. It is interesting to note that the result based on the second law of thermodynamics is the same as the one based on the first law, but with the additional assumption of isothermal processes.

Upon combining the above results with the relations (16) and (17) and integrating over a period, we obtain the dissipated energy per unit volume in a cycle,

$$W_d = \int \sigma \,\mathrm{d}\varepsilon = \int_0^{2\pi/\omega} \sigma_0 \sin(\omega t+\delta)\varepsilon_0\omega \cos \omega t \,\mathrm{d}t,$$

and upon performing the integration,

$$W_d = \sigma_0\varepsilon_0\pi \sin \delta. \tag{20}$$

Hence the tangent of the phase angle δ, known as the loss tangent, is

$$\tan \delta = \frac{W_d}{\pi \sigma_0 \varepsilon_0 \cos \delta}.$$

The strain attains its maximum value ε_0 at $t = T/4$, and the corresponding value of the stress is $\sigma = \sigma_0 \cos \delta$. If W_s denotes the stored energy per unit volume of a linear elastic material strained up to ε_0 with corresponding stress $\sigma = \sigma_0 \cos \delta$, then W_s is

$$W_s = 1/2 \sigma_0 \varepsilon_0 \cos \delta,$$

so that

$$\tan \delta = \frac{W_d}{2\pi W_s}. \tag{21}$$

3.2. *Comparison with Kelvin model*

The analysis presented above may be compared with that used frequently in structural dynamics. Assume that a body or an element is subjected to a displacement-controlled sinusoidal excitation given by $\Delta = \Delta_0 \sin \omega t$. The resulting steady-state force is

$$F = F_0 \sin(\omega t + \delta).$$

As a rule the Kelvin model is used in structural dynamics, though without being explicitly referred to: it is commonly known by other names, such as "viscous model" or "standard structural model", and it is sometimes thought, confusingly, that all linear damping models are Kelvin models. With the Kelvin model, the force displacement relation is $F = K\Delta + H\dot{\Delta}$, where K and H are the stiffness and viscous coefficient, respectively, of the element under consideration. By comparing the two expressions for the force, we have the result that $\cos \delta = K\Delta_0/F_0$ and $\sin \delta = H\omega\Delta_0/F_0$. Hence the loss tangent is

$$\tan \delta = \frac{H\omega}{K}. \tag{22}$$

With the help of this expression, the force becomes

$$F = F\Delta + \frac{K}{\omega} \tan \delta \dot{\Delta}.$$

With the preceding expression for $\sin \delta$ and with eqn (20), the following expression is obtained for the energy dissipated in a complete cycle by the element under consideration:

$$W_d = \pi H \omega \Delta_0^2. \tag{23}$$

If the element comprises a single-degree-of-freedom system with mass M and natural frequency $\omega_0 = \sqrt{K/M}$, then the damping coefficient is $\xi = H/H_{crit} = H/2M\omega_0 = H\omega_0/2K$; hence the viscous coefficient is $H = 2M\omega_0\xi = (2K/\omega_0)\xi$. Comparing this expression for the one previously found for the viscous coefficient, $H = \tan \delta K/\omega$, yields the result

$$\xi = \frac{\tan \delta}{2} \frac{\omega_0}{\omega},$$

so that at resonance ($\omega = \omega_0$) the damping coefficient equals one-half of the loss tangent.

With the help of eqn (21) it is also found that

$$\xi = \frac{W_d}{4\pi W_s}$$

at resonance. From eqn (23) we observe that the dissipated energy as predicted by the Kelvin model is quadratic in the displacement and linear in the frequency, while from eqn (22) we note that the loss tangent is linear in the frequency. This predicted behavior is contrary to that observed in most structural materials (Nashif *et al.*, 1985; Kelly, 1991), in which the damping coefficient is essentially independent of frequency. This frequency independence is usually modeled by hysteretic damping (Kelly, 1991; Nashif *et al.*, 1985), which although convenient in frequency-domain analysis is intractable in time-domain analysis. The latter is often preferable because nonlinearities can then be taken into account.

3.3. *Frequency representation*

A mathematically powerful way of presenting the loss tangent is by means of the frequency representation which will be used here (Golden and Graham, 1988). Let ζ denote the function

$$\zeta(t) = J(0)\delta\delta(t) + \dot{J}(t)H(t), \tag{24}$$

where $H(t)$ is the Heaviside step function and $\delta(t)$ is the Dirac delta function. The strain may then be represented (Golden and Graham, 1988) as

$$\varepsilon(t) = \int_{-\infty}^{t} \zeta(t-t')\sigma(t')\,dt', \tag{25}$$

where the lower limit may be replaced by t_1 if $\varepsilon(t) = 0$ for $t < t_1$. Similarly, we may define

$$\mu(t) = G(0)\delta(t) + \dot{G}(t)H(t),$$

so that

$$\sigma(t) = \int_{-\infty}^{t} \mu(t-t')\varepsilon(t')\,dt'. \tag{26}$$

Let $\hat{\zeta}(\omega)$ denote the Fourier transform of $\zeta(t)$, i.e.

$$\hat{\zeta}(\omega) = \int_{0}^{t_r} \zeta(t)\,e^{-i\omega t}\,dt \stackrel{\text{def}}{=} \mathscr{F}\{\zeta\},$$

with similar definitions for $\hat{\mu}(\omega)$, $\hat{\sigma}(\omega)$ and $\hat{\varepsilon}(\omega)$. It can easily be shown that

$$\hat{\zeta}(\omega) = i\omega\bar{J}(i\omega), \quad \hat{\mu}(\omega) = i\omega\bar{G}(i\omega),$$

where $\bar{J}$ and $\bar{G}$ denote the Laplace transforms as before. By taking the Fourier transforms of eqns (25) and (26) we obtain

$$\hat{\varepsilon}(\omega) = \hat{\zeta}(\omega)\hat{\sigma}(\omega), \quad \hat{\sigma}(\omega) = \hat{\mu}(\omega)\hat{\varepsilon}(\omega),$$

so that

$$\hat{\mu}(\omega) = \frac{1}{\hat{\zeta}(\omega)}.$$

The complex functions $\hat{\mu}(\omega)$ and $\hat{\zeta}(\omega)$ are respectively known as the complex modulus and

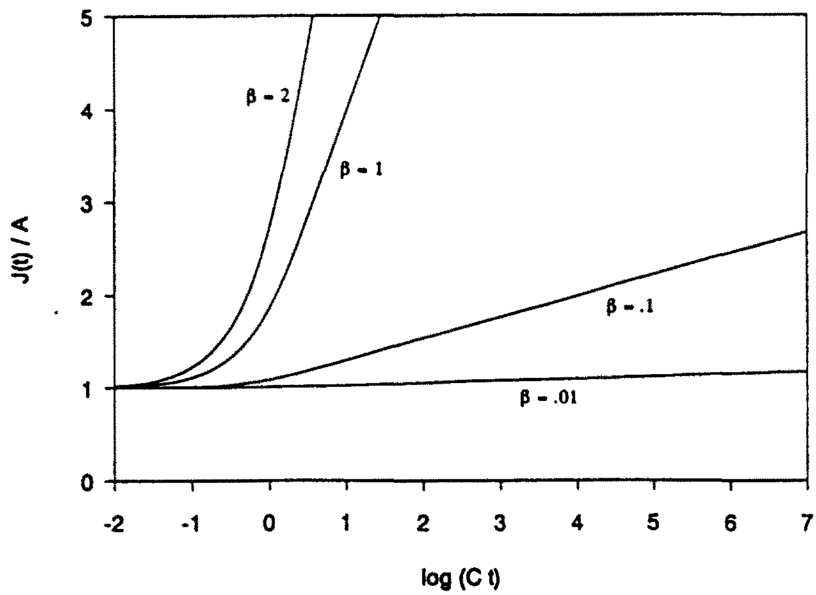

Fig. 2. Continuous modified Kuhn model: creep function for various values of β.

complex compliance. If $\hat{\mu}(\omega) = \hat{\mu}_1(\omega) + i\hat{\mu}_2(\omega)$, where $\mu_1(\omega)$ and $\mu_2(\omega)$ are usually referred to as the storage modulus and the loss modulus, respectively, then the loss tangent is

$$\tan\delta = \frac{\hat{\mu}_2(\omega)}{\hat{\mu}_1(\omega)}.$$

Similarly, if $\hat{\zeta}(\omega) = \hat{\zeta}_1(\omega) - i\hat{\zeta}_2(\omega)$, then

$$\tan\delta = \frac{\hat{\zeta}_2(\omega)}{\hat{\zeta}_1(\omega)}. \tag{27}$$

For the continuous modified Kuhn model it follows from eqn (11) that

$$\hat{\zeta}(\omega) = A + B\ln\left(1 + \frac{C}{i\omega}\right) = A + B\ln\sqrt{1 + \frac{C^2}{\omega^2}} - iB\tan^{-1}\left(\frac{C}{\omega}\right).$$

The loss tangent of the continuous modified Kuhn model is therefore

$$\tan\delta = \frac{\beta\tan^{-1}\dfrac{1}{q}}{1 + \beta\ln\left(1 + \dfrac{1}{q^2}\right)^{1/2}}, \tag{28}$$

where $\beta = B/A$ and $q = \omega/C$. Plots of the creep function $J(t)$, as given by eqn (7), against the logarithm of time and of the loss tangent, given by eqn (28), against the logarithm of q are shown in Figs 2 and 3, respectively. It is observed from the latter plot that for small values of β, the loss tangent is almost constant over a wide frequency range, which is a characteristic of structural materials, and the values are similar to those observed in structural materials. The Kelvin model notoriously fails to predict this type of behavior, since its loss tangent varies linearly with the frequency.

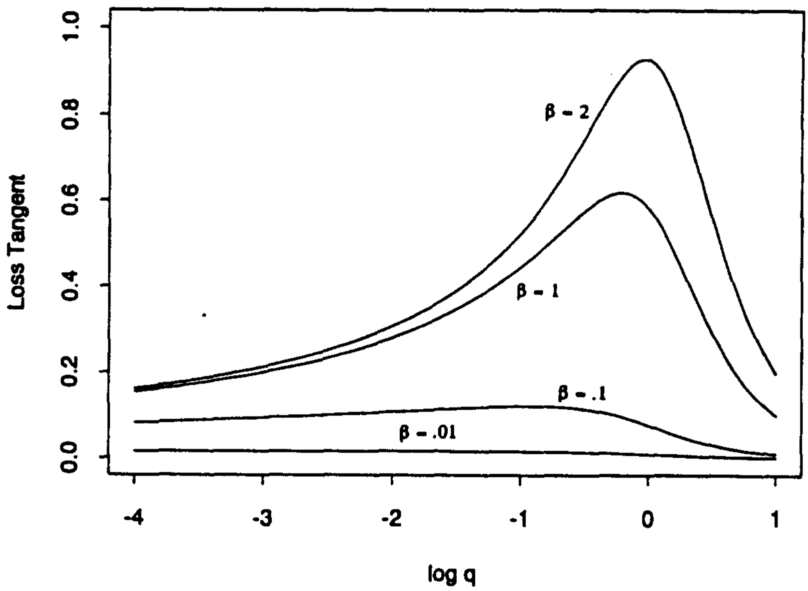

Fig. 3. Continuous modified Kuhn model: frequency dependence of loss tangent for various values of β.

4. THE DISCRETE MODIFIED KUHN MODEL

4.1. *Relation to continuous model*

Let $J_N(t)$ denote the following truncated Dirichlet (or Prony) series:

$$J_N(t) = A + B \ln r \sum_{m=0}^{N} (1 - e^{-Ct/r^m}), \tag{29}$$

where r is a parameter subject to $r > 1$. This series represents the creep function of a generalized Kelvin model, shown in Fig. 4, consisting of a spring with compliance A in series with $N+1$ Kelvin elements, each of which has the same spring compliance $B \ln r$, and whose retardation times form a discrete spectrum with values r^m/C, $m = 0, \ldots, N$. It will be shown that $J_N(t)$ has as a limit, under conditions that will be specified, the creep function $J(t)$ of the continuous modified Kuhn model. Consequently the model described by $J_N(t)$ may be called the *discrete modified Kuhn model*. It is noteworthy that the only parameters describing the model are A, B, r and N, so that the number of parameters is independent of N.

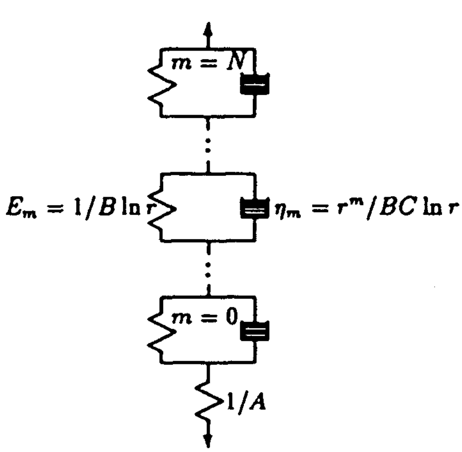

Fig. 4. Discrete modified Kuhn model: representation as a generalized Kelvin model.

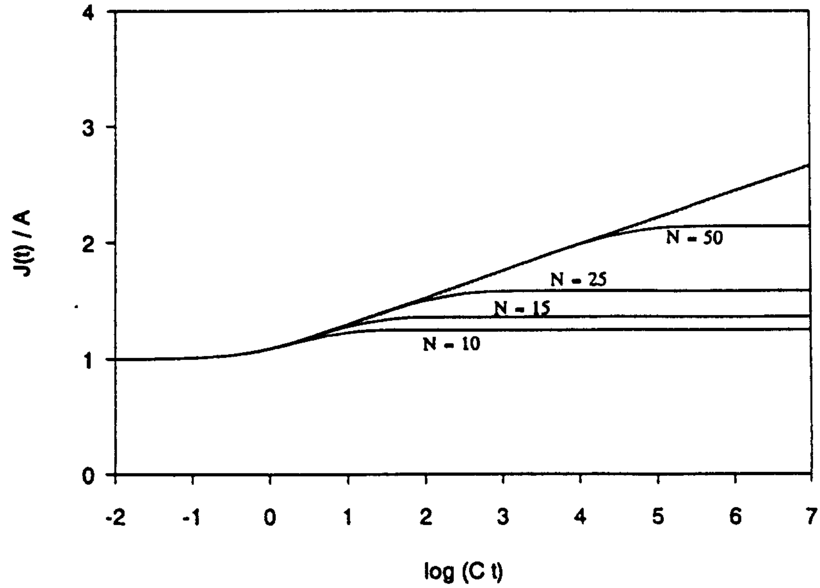

Fig. 5. Discrete modified Kuhn model: creep function for various values of N; $\beta = 0.1$, $r = 1.25$ (dotted line: continuous model).

Theorem

$$J_N(t) \to J(t) \quad \text{as} \quad N \to \infty, r \to 1+, r^N \to \infty.$$

It should be noted that the first two conditions, $N \to \infty$ and $r \to 1+$, are not sufficient for the third, $r^N \to \infty$, as can be seen from the following counter-example: If r is given by the sequence $1+1/N$, then it is seen that $r \to 1+$ as $N \to \infty$, while $r^N = (1+1/N)^N \to e$.

Proof. The improper integral of expression (8) for the creep function is given by

$$I \stackrel{\text{def}}{=} \int_{1/C}^{\infty} (1-e^{-t/\tau})\frac{d\tau}{\tau} = \lim_{H\to\infty}\int_{1/C}^{H}(1-e^{-t/\tau})\frac{d\tau}{\tau}.$$

By definition of the Riemann integral, the integral I is

$$I = \lim_{H\to\infty} \lim_{\substack{N\to\infty \\ \|\Delta\tau_m\|\to 0}} \sum_{m=0}^{N} f(\tau_m^*)\Delta\tau_m,$$

where $f(\tau) \stackrel{\text{def}}{=} (1-e^{-t/\tau})/\tau$, τ_m^* stands for an arbitrary value inside the interval $\tau_m \leqslant \tau_m^* \leqslant \tau_{m+1}$, $\Delta\tau_m = \tau_{m+1}-\tau_m$, and N is the number of intervals. It is noted that τ_0 is equal to $1/C$, i.e. it is the lower limit of integration, and that $H = r^N/C$, hence $H \to \infty$ when $r^N \to \infty$. If we now further take $\tau_m^* = \tau_m$, the above equation becomes

$$\int_{1/C}^{\infty}(1-e^{-t/\tau})\frac{d\tau}{\tau} = \lim_{r^N\to\infty}\lim_{\substack{N\to\infty \\ r\to 1}}\sum_{m=0}^{N}(1-e^{-Ct/r^m})(r-1). \tag{30}$$

If the function $\ln r$ is expanded in a Taylor series about $r = 1$, we obtain

$$\ln r = 0+(r-1)-\tfrac{1}{2}(r-1)^2+O((r-1)^3),$$

or

$$\ln r = (r-1)+O((r-1)^2).$$

Then

$$\lim_{\substack{r^N \to \infty \\ N\to\infty, r\to 1}} \ln r \sum_{m=0}^{N} (1-e^{-Ct/r^m}) = \lim_{\substack{r^N \to \infty \\ N\to\infty, r\to 1}} [(r-1)+O(r-1)^2] \sum_{m=0}^{N} (1-e^{-Ct/r^m})$$

$$= \lim_{\substack{r^N \to \infty \\ N\to\infty, r\to 1}} (r-1) \sum_{m=0}^{N} (1-e^{-Ct/r^m}).$$

This relation, with the help of eqn (30), proves the theorem.

4.2. *Properties of the discrete model*

The loss tangent of the discrete modified Kuhn model can be derived with the help of relation (27). By applying relation (24) to eqn (29), we obtain

$$\zeta(t) = A\delta(t) + BC \ln r \sum_{m=0}^{N} \frac{1}{r^m} e^{-Ct/r^m} H(t),$$

so that the complex compliance is

$$\hat{\zeta}(\omega) = A + BC \ln r \sum_{m=0}^{N} \frac{1}{r^m} \mathscr{F}\{e^{-Ct/r^m}\}$$

$$= A + BC \ln r \sum_{m=0}^{N} \left(\frac{C}{C^2+\omega^2 r^{2m}} - i \frac{\omega r^m}{C^2+\omega^2 r^{2m}} \right).$$

Consequently, with $\beta = B/A$ and $q = \omega/C$ as before, the loss tangent is given by

$$\tan\delta = \frac{\hat{\zeta}_2(\omega)}{\hat{\zeta}_1(\omega)} = \frac{\sum_{m=0}^{N} \frac{q r^m}{1+q^2 r^{2m}}}{\frac{1}{\beta \ln r} + \sum_{m=0}^{N} \frac{1}{1+q^2 r^{2m}}}.$$

Plots of the creep function and of the loss tangent, shown in Figs 5 and 6, respectively, confirm the convergence to the continuous modified Kuhn model with increasing N; in the

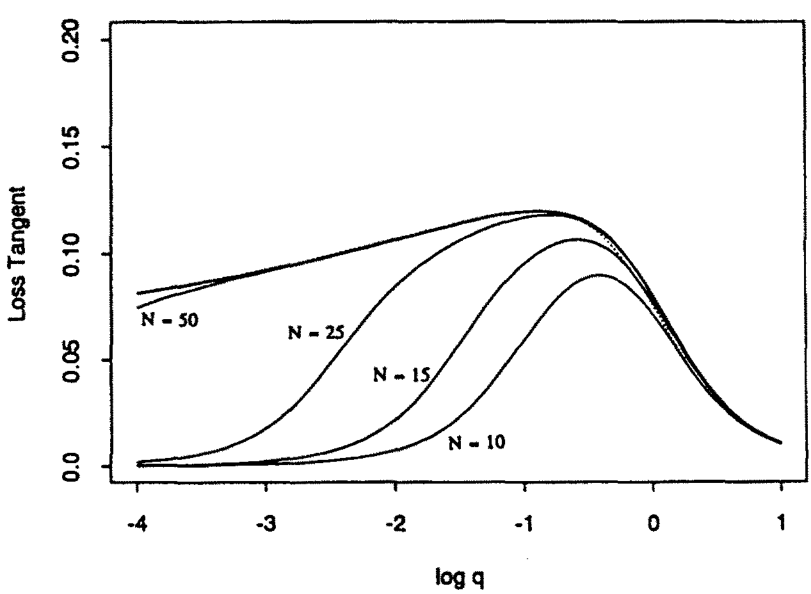

Fig. 6. Discrete modified Kuhn model: frequency dependence of loss tangent for various values of N; $\beta = 0.1$, $r = 1.25$ (dotted line: continuous model).

range shown, the curves corresponding to $N \geqslant 100$ are virtually indistinguishable from those for the continuous model.

An important feature of the discrete model is the fact that it is an internal-variable model. For any stress history $\sigma(t)$ such that $\sigma(-\infty) = 0$, the strain $\varepsilon(t)$ may be writen as

$$\varepsilon(t) = \int_{-\infty}^{t} J_N(t-t')\,\mathrm{d}\sigma(t')$$
$$= \varepsilon^{e}(t) + \sum_{m=0}^{N} \alpha_m(t),$$

where

$$\varepsilon^{e}(t) = A\sigma, \quad \alpha_m(t) = B \ln r \int_{-\infty}^{t} (1 - \mathrm{e}^{-C(t-t')/r^m})\,\mathrm{d}\sigma(t').$$

It can easily be seen that each $\alpha_m(t)$ obeys the rate equation

$$\dot{\alpha} + \frac{C}{r^m}\alpha = \frac{BC \ln r}{r^m}\sigma$$

corresponding to the Kelvin model. The discrete model can be extended into the nonlinear range by replacing the right-hand side of the rate equation with a nonlinear function of σ; for example, the factor σ can be replaced, in accordance with rate-process theory, by $D \sinh(\sigma/D)$, where D is a constant.

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